ics are of an analytical not a synthetical character. This opinion has not only received the support of Leibnitz as well as of the school of philosophy to whom the truths of mathematics—especially those of Arithmetic and Algebra—represent merely verbal propositions, but it has been peculiarly strengthened by recent mathematical developments. If we consult the writings of those who have maintained the opposite doctrine, viz., the synthetical character of mathematical truths we shall find that they maintain this only of certain of the fundamental truths employed in mathematics. Thus the axioms proper of Euclid are admitted by Kant and Mansel to be analytical. When they seek for an example of a synthetical truth, they find it in the fifth and sixth postulates, sometimes enumerated as the eleventh and twelfth axioms. But it is precisely such principles as the latter which are now regarded as expressing not universal and necessary principles of all geometry but only the particular and contingent properties of the space with which we are acquainted ; and of which Clifford asserts "for all we know, any or all of them may be false." If now by mathematical inference or reasoning be understood the form of reasoning common to all mathematical thought, there seems to be left as a residuum, only those processes of analytical inference which are expounded in the ordinary formal or Aristotelian logic.

It is possible to support this view, to regard those principles which are really synthetic and fertile in mathematics, as either gathered from actual experience, or as hypothetically assumed in regard to some possible experience; and, on the other hand to regard the process by which these fundamental assumptions are worked out into their consequences as purely syllogistic. The necessity with which these consequences flowed would then be strictly formal and logical. This view would appear to be in exact accordance with the general principles laid down by J. S. Mill on the subject of demonstration and necessary truths (Logic, Book II, Chapter VI). It differs from Mill's view, only in not regarding the axioms proper as inductions from experience, and in extending the postulates to embrace those possibilities of Non-Euclidean geometry which Mill did not contemplate. Notwithstanding some criticism of Mill, this I understand to be the view actually adopted by Clifford. A theory more completely agreeing with Mill's principles is put forward by Erdmann in his work 'Die Axiome der Geometrie' resting his conclusions on the investigations of Riemann and Helmholtz.

It is to be observed however that this doctrine is in complete opposition to the theory contained in the quotation from James Mill with which we started. On that theory the rich content of mathematics could be evolved by a series of analytical or verbal transformations. J. S. Mill in the chapter of the Logic from which we have quoted, clearly shows the impossibility of such a view. Mathematical inference leads to new truths, and new truths, according to Mill, can only be reached if inference be impregnated by experience. Hence Mill held that axioms and postulates come from this source.

It still remains to be asked, Can Formal Logic, Syllogism or mere verbal inference, perform the attenuated task left to it, viz., the inferring process ?

Can verbal propositions, if no longer competent to give a new content, nevertheless be the means of passing from one content to another? At first sight, the great resemblance between the elementary propositions of mathematics and the propositions of Formal Logic, seems to favour this view. When in Arithmetic we form the judgment 2+1=3, or in Geometry discover the quantitative equivalence of the three angles of a triangle to two right angles, the resemblance to such purely analytical relations as are formulated by the *dictum de omni et nullo* is very great. In fact, our theory reduces predication itself to the equation of groups. Nevertheless, this resemblance is, I believe illusory, and so far is mere verbal inference from being able to perform the function, which James Mill allotted to it, that it cannot even perform the more modest task reserved to it by J. S. Mill.

It has long been a matter of observation that inferences exist, which while perfectly rigorous, yet do not admit of syllogistic analysis. The argument *a fortiori* is an example, and many others are furnished by what has been called the logic of relatives. These are examples of what older logicians called material consequence. The device by which Mansel and others have tried to reduce them to syllogistic form, is, as De Morgan truly says, an evasion. Both kinds of inference are found in mathematics but that inference which is most peculiarly mathematical is of the second kind, which escapes or defies the analysis of Formal Logic. Mill has exhibited Euclid I, 5 in syllogistic form but the reduction of the reasoning to this form is purely external. The force of mathematical reasoning is independent of the reduction. This is already implied in Dugald Stewart's remark, referred to by Mill, that it is not necessary to our seeing the conclusiveness of the proof in mathematical reasonings, that the axioms should be expressly adverted to. The same thing is conceded, perhaps unwittingly by Hamilton. "Mathematical, like all other reasoning," he says, "is syllogistic; but here *the perspicuous necessity of the matter necessitates the correctness of the form:* we cannot reason wrong."

If, now, we have reason to believe, that there exists in mathematical inference, a "necessity of the matter" existing in itself, and not merely derivative from the logical form, the question arises: Can we isolate it for itself? If we can do this we shall have grounds for concluding, that what is really fruitful in mathematics, is not, as has been so often supposed, initial definitions, axioms, etc., which we afterwards logically analyze and develop, but a certain synthetic mode of inference not identical with the analytical inference of formal logic, but distinct from it, *sui generis*, and perhaps opposed in character.

Before passing to the consideration of this synthetic and material necessity, it may be well to point out, that the whole controversy as it has hitherto existed between the *a priori* and *a posteriori* schools, between the Kantian and the empiricist, becomes for us irrelevant. In a paper in Mind in 1884 I pointed out that the Kantian theory does not explain the synthesis in mathematical truths. It only places the synthesis finished and complete, in the subject. Clifford puts the same point in another way when he says that the Kantian theory

makes the general statements of mathematics into particular statements. On the other hand the empirical school makes no attempt to explain the necessity of the consequence in itself. It is this which we are seeking to isolate. When this isolation has been effected, it will appear that mathematical inference is so related to ordinary analytical reasoning, as to stand in need neither of the Kantian nor experience hypothesis, neither of *a priori* form nor of inseparable association, in order that its necessary and universal character may be accounted for.

The idea has sometimes been entertained, that logical analysis has reached through the calculus contained in Boole's Laws of Thought this more penetrating character and that here we have an instrument which can make inferences beyond the range of formal logic. This is not so. Mr. A. J. Ellis has shown that it really does less; and in the papers of Leslie Ellis it is pointed out that for dealing with those forms of collateral inferences which we find in tha logic of relatives it is as ineffectual as formal logic.

The immense suggestiveness of Boole's work lies in the circumstance that he has reduced formal logic to a calculus and that logical doctrines are put in a form in which they suggest mathematical analogues. But the suggestion is, for the most part one of opposition. The things are brought into the same plane and thereby their opposition becomes apparent. This is precisely what from the foregoing we should expect. If, now, taking Boole's Laws of Thought as an exposition of formal logic in mathematical form, we ask the question 'Can we find within the area of mathematics any calculus presenting that antithetic but complementary character which a form of synthetic inference should present, as contrasted with a form of analytical inference?' I think we may answer 'Yes.' If we compare the fundamental equations of Boole's Laws of Thought with the equations which characterize some of those forms of multiplication discussed by Grassmann, and employed by him in his 'Ausdehnungslehre' we seem to find the antithesis which we seek. Already in the principles of the Differential Calculus this antithesis was to be found, and by an intuition of genius was perceived by Boole. For the purpose of this paper we shall confine ourselves to the following equations.

In Boole's Laws of Thought we find

$$x^2 = x \qquad (1).$$

$$x(1-x) = 0 \qquad (2).$$

In Grassmann's we find

$$e_p{}^2 = 0 \qquad (3).$$

$$e_p e_s = -e_s e_p \qquad (4).$$

If we compare these two sets of equations we shall find that they differ in this fundamental characteristic, that whereas the first set of equations belong to an algebra of self-identical unrelated units the second set belong to an algebra in which relation, synthesis, references beyond self, is essential.

The equations (3) and (4) have given rise to some difference of opinion in

regard to their mutual relations. H. Hankel, H. J. S. Smith and Mr. Whitehead assert that the first of these equations follows necessarily from the second. Buchheim however points out that this is not the case, that (4) does not involve (3), and Clifford regards (4) as following from (3); that is, we may assume (4), without asserting (3), but not (3) without (4).

In this Clifford and Buchheim seem to be right. The equation $r_s r_p = -r_p r_s$ proves $r_s^2 = 0$ (in the case where $p = s$) if r_s and r_p exist only as necessary relatives which become equal to 0 when the difference and therefore the relation between them disappears. If they exist as such, equation (4) seems to follow, but equation (3) follows from this relativity, not from (4), which may be affirmed even when the factors involved exist out of and apart from their mutual relation. The truth seems to be that equations of this type of algebra may participate in characters derived from equations of the Boole type, and so give rise to hybrid species which may be useful for particular interpretations. In general we may distinguish (1) the purely logical algebra whose relations only of identity and difference, coincidence and exclusion are admitted. (2) The algebra of material consequence where the units are not indifferent but where the *entia* involved are essentially relative and the algebra consequently synthetic.

Both these algebras lead to a final equation in which each passes over into the opposite. Boole's $x(1-x)=0$ is in his system the ultimate condition of logical interpretability but it also expresses, the one relation into which logical terms enter with each other, and the equation $i_s^2 = \pm 1$ (Clifford) expresses the disappearance of relativity and the return of the merely logical relations of difference and identity.

Ordinary Arithmetic, Algebra, and Metrical Geometry are a combination of these two kinds of Algebra. Up to a certain point, the principles of both can be alternately applied, but at one point the necessity arises for a deeper fusion. The introduction of relation into a logical calculus involves, as has been pointed out by Mr. Venn, the very thing which Boole excludes—the admission of exponents. Conversely, in ordinary mathematics the appearance of exponents involves essential relation. In the sign $\pm\sqrt{\ }$ the alternation of the purely logical and the relational aspects is still continued, and the same is the case in the Differential Calculus. But in the latter, and in imaginary expressions this alternation of independent aspects ceases. It the calculus the externality of the logical consideration ceases at infinity. In the imaginary it ceases in the immediate combination of the signs + and −.

In a paper on the Imaginary of Logic (British Association, 1898) I put forward the view that as the square root of a positive quantity is + *or* − the square root of a negative quantity may be expected to be + *and* −, in view of the logical relation between '*and*' and '*or*' pointed out originally by De Morgan, and subsequently, and independently by Schroeder. This theory is the opposite of one put forward in an early number of the Cambridge and Dublin Mathematical Journal by Gregory, viz., That the signs + and − are themselves the subject of the exponential operation. The object of the paper was to show that

a necessary relation of the signs + and − as affecting the factors respectively was the essential characteristic of imaginary quantities. The essence of this relation would then coincide with Boole's $x(1-x)$. The subsequent portion of the paper sought to verify this theory throughout the various geometrical interpretations which imaginaries have received.

Since the paper was written I have found that its conclusions receive support from a remarkable series of papers by Mr. A. B. Kempe F. R. S. Mr. Kempe has shown that between the mathematical theory of points and the logical theory of statements, a striking correspondence exists. Between the laws defining the form of a system of points, and those defining the form of a system of statements, perfect sameness exists with one exception. The former is subject to a law to which the latter is not subject. It is sufficient here to say that it is the law "which expresses the fact that two straight lines can only cut once."

From these conclusions we may draw the converse inference, that the laws which govern geometrical theory can be deduced from logical or purely analytical principles, taken in conjunction with that law in which the form of a system of points differs from the form of a system of statements. We have now to ask, Is there anything omitted from the form of a system of statements as contemplated by Mr. Kempe, or by the ordinary logic (and there is complete agreement between them) which would account for the absence of the particular law which distinguishes geometrical theory? I think there is. Mr. Kempe in order to effect his assimilation of the logical to the geometrical theory, and in particular in explaining the processes of immediate inference has introduced two constants which play the same part in the logical theory that the 'absolute' does in geometry. He entitles them 'truism' and 'falsism' respectively. It is by relation to these that such logical relations as contrariety, sub-contrariety, sub-alternation analogous to the metrical relations of points in geometry are determined. He considers "truisms" and "falsisms" as propositions or statements standing in the system of statements on the same footing with all other statements. In reality this is not so. The truism and falsism of Mr. Kempe are really the laws of Identity and Contradiction in disguise, and every synthetic statement or proposition expresses more than what these laws require. The principle that a real proposition refers to, or is a synthesis with, something more than itself, is as old as Aquinas, and is indeed the fundamental principle which makes our thinking dependent on experience (Cf. Bradley's Principles of Logic). It is the non-recognition of this which prevents Mr. Kempe from evolving the relation of non-collinearity from the relation of a truism and falsism to each other which ought to be capable of being done, if it were true that these propositions could rank *pari passu* with all other propositions. A truism is not as such a true proposition. Apart from the postulate of synthesis no logical relation exists between the truism and falsism. Contradictories are in this case compatible as Venn and Kant before him have pointed out.

If these views be true I believe it to be possible to deduce the properties of Euclidean space, not from the analytical laws of thought, but from the pure

postulate of synthesis, when subjected to conditions arising from these laws. The postulate can be shown to involve two things (1) Infinity, (2) the necessary relation or connection of what Mr. Kempe styles truism and falsism equivalent to Boole's $x(1-x)=0$.

It remains to point out the connection which exists between the logical 'absolute' of Mr. Kempe and the theory of the imaginary referred to in the course of this paper. I was led to that analysis from consideration of the correspondence between the logical relation of a copulative or conjunctive to a disjunctive proposition, and the mathematical relation of imaginary roots to the roots of positive quantities. A similar relation has been perceived by Mr. Kempe. "The symmetrical resultant of the triad [of statements] a, b, f [f=falsism] is the statement usually written, a and b, and the symmetrical resultant of the triad, a, b, t [t=truism] is the statement usually written a or b." If the relation of truism and falsism or in Boole's language $x(1-x)$ be, as we assert the essence of the mathematical imaginary; and if the same constants have in Mr. Kempe's analysis disclosed themselves as the essence of the geemetrical 'absolute' a deeplying relation is revealed between the methods of metrical and projective Geometry.

Finally, there exists a curious analogy between the geometrical, and certain theories of the metaphysical absolute. The temptation lies near at hand to evolve the synthetical or given element out of the laws of Identity and Contradiction. Fichte's evolution of the Non-Ego out of the Ego is effected in this way and thus arises his theory of the absolute Ego. Precisely the some error is committed, if an attempt be made to dispense with the postulate of synthesis, with the *given*, and to evolve mathematics out of analytical propositions. Mr. Kempe comes near this mistake when he treats truisms and falsisms as propositions on a line with all other propositions.

It remains to draw the final conclusions of this paper. The fertile propositions of mathematics from which its wealth of content and the treasures of mathematical knowledge are drawn, are not synthetical in the sense in which Kant and the Empiricists alike maintain them to be, viz., that the truths pre-exist and are thus seen to be synthetical, the synthetical character being as it were something subsequent to the content of the proposition and attaching to it as it were adjectivally; but in this sense that those propositions are themselves the product of pure synthesis, that the very possibility of advance from entity to entity or unit to unit, or relation to correlate, determines all those laws which mathematics is employed in exploring and tracing into all their consequences, and which are infinitely more fruitful than the analytical laws of Formal Logic or the Calculus of classes and statements. Pure synthesis generally is that "necessity of the matter" of which Hamilton spoke, the principal of material consequences, which characterizes every genuine department of mathematics and defies further logical analysis.

Fancourt, Balbriggan, Ireland, 10 August, 1899.

EXAMPLES OF A FEW ELEMENTARY GROUPS.*

BY DR. G. A. MILLER.

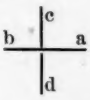
It is easy to verify that the following four substitutions,

$$1 \qquad ab \qquad cd \qquad ab\,cd$$

constitute a substitution group, which has the following properties: (1) The product of any two of these substitutions of order two is the third, and the product of the three is identity; (2) These products are independent of the order of the factors, *i. e.*, all the substitutions are commutative;† (3) The smallest group that contains two of these substitutions of order two must also contain the third, *i. e.*, the group is generated by any two of its substitutions of order two, but it is not generated by one of them; (4) The group contains three subgroups of order two and one of order one.

We proceed to give several geometric illustrations of this group.

Representing the positive half of the x-axis by a, the negative half by b, the positive half of the y-axis by c, and the negative half by d, we observe that the rotation of the plane around the y-axis through an angle of 180° corresponds to the substitution ab, and the rotation around the x-axis through 180° corresponds to cd. The effect obtained by these two rotations, in succession, is clearly equivalent to a rotation of the plane through 180° on the origin as a pivot. This operation is also of order two and it corresponds to $ab.cd$.



Since the law of combination of these three operations is exactly the same as that of the given substitutions we say that these three operations and identity constitute a group which is *simply isomorphic* to the given substitution group. Representing the given operations analytically we obtain the following equations:

$$ab\begin{cases} x'=-x \\ y'=y \end{cases} \qquad cd\begin{cases} x'=x \\ y'=-y \end{cases} \qquad ab.cd\begin{cases} x'=-x \\ y'=-y \end{cases}$$

It may be observed that the third of these operations is equivalent to a rotation of space around the z-axis through 180°.

We may obtain another geometric illustration of the same group by considering the inversion of the plane with respect to any circle of radius k and the rotation of the plane through an angle of 180° around a line passing through the

*In this article we shall not presuppose any knowledge of the theory of groups except the facts which were developed in our article in the November number of this Journal.

†If every two substitutions of a group are commutative the group is said to be Abelian. Hence this group of order four is an Abelian group. The group of order six which is given in the said article of the November number of this Journal is non-Abelian.

center of the circle. Since each of these operations is of order two and since they are commutative their product must be of order two and it must be commutative with each of these two operations. For, if s_1, s_2 represent two different commutative operations of order two we have

$$(s_1 s_2)^2 = s_1 s_2 s_1 s_2 = s_1^2 s_2^2 = 1.$$

This proves that $s_1 s_2$ is of order two. From $s_1 s_2 s_1 = s_1 s_1 s_2$ we observe that s_1 is commutative with $s_1 s_2$. Similarly we see that s_2 is commutative with $s_1 s_2$. Hence we observe that *any two different commutative operations of order two must generate a group which is simply isomorphic to the given group of order four.* If we use the given line as the x-axis and the perpendicular to it through the center of the circle as the y-axis we may represent the given operations analytically as follows :

$$ab\begin{cases} x' = \dfrac{k^2 x}{x^2+y^2} \\ y' = \dfrac{k^2 y}{x^2+y^2} \end{cases} \qquad cd\begin{cases} x' = x \\ y' = -y \end{cases} \qquad ab.cd\begin{cases} x' = \dfrac{k^2 x}{x^2+y^2} \\ y' = -\dfrac{k^2 y}{x^2+y^2} \end{cases}$$

To verify analytically that the last one of these operations is of order two we let

$$x'' = \frac{k^2 x'}{x'^2 + y'^2} = k^4 \frac{x}{x^2+y^2} \frac{(x^2+y^2)^2}{k^4 (x^2+y^2)} = x,$$

$$y'' = -\frac{k^2 y'}{x'^2 + y'^2} = k^4 \frac{y}{x^2+y^2} \frac{(x^2+y^2)^2}{k^4 (x^2+y^2)} = y.$$

We have now given two geometric illustrations of the given group of order four† and we observed that the characteristic property of this group is, that it is generated by two different commutative operations of order two. There is another substitution group of order four whose characteristic property is entirely different. The substitutions of this group are

$$1 \qquad ac.bd \qquad abcd \qquad adcb.$$

Each of the last two substitutions is the third power of the other and each one of these generates the entire group ; *i. e.*, the smallest group that contains one of these substitutions must contain all the substitutions of this group of order four. It can readily be verified that these four substitutions obey the same com-

*These substitutions may be obtained by representing the segment of the axis which is outside the circle by a, the segment within the circle by b, the upper semi-circumference by c, and the lower semi-circumference by d.

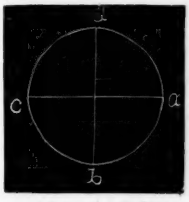
†In the latter example we could have inverted space with respect to a sphere intead of inverting the plane with respect to a circle.

binatory laws as the numbers which are written below them in the following arrangement:

1	$ac.bd$	$abcd$	$adcb$
1	-1	$\sqrt{-1}$	$-\sqrt{-1}$

The last two numbers can evidently be interchanged without affecting the laws of combination, but none of the other numbers permit such an interchange.

If we denote the points where two perpendicular diameters meet a circle by a, b, c, d, we observe that the substitution $abcd$ is equivalent to rotating this circle on its center through 90°, $ac.bd$ is equivalent to a rotation through 180°, and $adcb$ is equivalent to a rotation through 270°, or through $-90°$. The characteristic property of this group is that it is generated by an operator* of order four. When a group is generated by a single operator of order n it is called the *cyclical* group of order n. It should be observed that the cyclical group of order four contains only one subgroup of order two, viz., the one which corresponds to the rotations through 180° and 360°, while the given non-cyclical group of this order contains three such subgroups.



We have now considered two groups of order four whose combinatory laws are different, *i. e.*, two groups which are not simply isomorphic. Such groups are said to be distinct *abstract* groups. Two groups which are simply isomorphic are said to be the same abstract group, regardless of the notation by means of which they may be represented, *e. g.*, 1, ab, cd, $ab.cd$, and 1, $ab.cd$, $ac.bd$, $ad.bc$ are different as substitution groups but they represent the same abstract group since the law of combination of their substitutions is the same. We may state without proof that there are only two abstract groups of order four; *i. e.*, If four operators form a group their laws of combination must be the same as those of one of the given groups of order four.

We proceed to give a geometric illustration of the group of order six which is composed of all the substitutions that can be formed with three letters.† The substitutions of this group are

1	abc	acb	ab	ac	bc

Dividing the circle into three equal parts and drawing diameters through these points of division, we observe that abc and acb correspond to rotations of the circle on its center through 120° and 240°, respectively; ab, ac, and bc corres-

*The substitutions of a group represent operators as well as the result of operations. The group elements may therefore be called operations or operators. It is necessay to distinguish between the group elements and the elements of the substitutions, the former term is frequently used to denote the operators of a group since the group is really composed of these operators as elementary parts. When the word "element" is used in connection with a group we have sometimes to decide from the context whether it means an operator or a letter of the substitutions of the group.

†The group which is composed of all the possible substitutions of degree n is called the *symmetric* group of degree n. It is of order $n!$. Cf. THE AMERICAN MATHEMATICAL MONTHLY, Vol. VI, page 257.

pond to the rotations of the plane through 180° around the diameters going through c, b, and a, respectively. If we perform any two of these rotations in succession the result is equivalent to a single rotation which corresponds to the product of the two substitutions corresponding to the two rotations and taken in the same order ; *e. g.*, the rotation on the center through 240°, followed by rotation through 180° around the diameter through a is equivalent to the single rotation through 180° on the diameter through c, since $acb.bc=ab$. This result should also be seen geometrically.

All the products given in Vol. VI, page 256, of this Journal, may be directly verified by means of the last figure. The six rotations which correspond to the substitutions of the symmetric group of degree three are thus seen to form a very interesting group of rotations according to which the plane (or space) may be transformed. The determination of all the possible groups of motion by means of which space may be transformed forms a very interesting problem in the theory of groups, which was first studied by Camille Jordan, *Annali di Matematiche*, 1868, Vol. 2, page 167. The group of finite rotations are given in somewhat greater details in Klein's *Ikosaeder*, 1884, Chapter I.

Another important illustration of the symmetric group of three elements is furnished by the six anharmonic ratios of four points. These ratios may be placed in six different ways in a 1, 1 correspondence with the substitutions of this symmetric group. One of these ways is as follows :

1	abc	acb	ab	ac	bc
$\lambda,\ \lambda$	$\lambda,\ \frac{\lambda-1}{\lambda}$	$\lambda,\ \frac{1}{1-\lambda}$	$\lambda,\ 1-\lambda$*	$\lambda,\ \frac{1}{\lambda}$	$\lambda,\ \frac{\lambda}{\lambda-1}$

where the notation $\lambda, \frac{\lambda-1}{\lambda}$ means that λ is to be replaced by $\frac{\lambda-1}{\lambda}$. *E. g.*, performing the third and fourth operation in succession, we have $\lambda, \frac{1}{1-(1-\lambda)} = \lambda,\ 1/\lambda$ just as $acb.ab=ac$; performing the fourth and third operation in succession, we have $\lambda, 1-\frac{1}{1-\lambda}=\lambda,\ \frac{\lambda}{\lambda-1}$ just as $ab.acb=bc$, etc. For other illustrations of this group the reader may consult Burnside's *Theory of Groups*, 1897, page 18.

The symmetric group of degree four contains 24 substitutions. These correspond to the 24 rotations which transform a cube into itself, for these rotations permute the four diagonals of the cube in every possible manner. The axes of rotation are the lines which join the middle points of the opposite faces, those which join the middle points of the opposite edges, and the diagonals. There are three axes of the first kind and we may rotate the cube around one of these

*It may be observed that $(\lambda, 1-\lambda)^2=1$ while $(\lambda, \lambda-1)^\alpha=\lambda, \lambda-\alpha$; *i. e.*, the first of these two substitutions is of order two while the second does not have a finite order.

axes through an angle of 90°, 180°, 270°, or 360° so that after each rotation the entire cube occupies the same space as it did before the rotation. Hence the symmetric group of degree four contains three cyclical subgroups of order four. Each of these contains a subgroup of order two and no two of these subgroups of order two are identical.

There are six axes of the second kind and we may rotate the cube into itself around one of these axes through 180° or 360°. As the corresponding subgroups of order two are different from the three given above we observe that the symmetric group of degree four contains nine subgroups of order two. The rotations around the diagonals correspond to the four snbgroups of order three that are contained in the symmetric group of degree four. We have now employed all the possible rotations which transform the cube into itself without changing its center, and have seen that the corresponding permutations of the diagonals give all the possible substitutions that can be formed with four elements. These 24 rotations constitute an interesting group of motion.

It is easy to see that the different powers of a circular substitution of degree n $(a_1 a_2 a_3 \ldots\ldots a_{n-1} a_n)$ constitute a group of order n. When $n=3$ we have the substitutions 1, $a_1 a_2 a_3$, $a_1 a_3 a_2$, and when $n=4$, the substitutions 1, $a_1 a_3$. $a_2 a_4$, $a_1 a_2 a_3 a_4$, $a_1 a_4 a_3 a_2$. These groups have been considered. In general we may divide the circumference of a circle into n equal parts and represent the points of division by a_1, a_2, a_3, ..., a_n. The n different positive rotations around the center of the circle through angles which are divisible by $2\pi/n$ will clearly constitute a group of operations that is simply isomorphic to the substitution group generated by the given circular substitution. Since the equation $x^n - 1 = 0$ has primitive roots all the roots of this equation constitute a group which is simply isomorphic with the cyclical group of order n.

From the preceding examples it may be inferred that the same group may present itself in many different forms as well as in different branches of mathematics. The fundamental group concept is that there is a system of operations (substitutions, rotations, complex numbers, etc.) such that the product of any two of them and the square of any one are again in the system. This necessary condition is not always a sufficient condition that a system of operations may constitute a group, but many operations, such as substitutions, obey *per se* the other necessary conditions.*

*Cf. Burnside, *Theory of Groups of a Finite Order*, page 11, or Weber's *Lehrbuch der Algebra*, Vol. 2, page 2.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

120. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

How many balls 1 inch in diameter can be put in a cubical box 1 foot in the *clear* each way, putting in the maximum number? [From Greenleaf's *Treatise on Algebra.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and the PROPOSER.

First, put in two layers of 144 balls each, then fourteen layers of 121 and 144 balls in alternate layers.

This gives nine layers of 144 each=1296
seven layers of 121 each= 847

Total=2143

Every four balls in 144 layer makes with each ball in a 121 layer a square pyramid, base 1 inch, and slant edge 1 inch.

$\therefore$ Altitude=$\frac{1}{2}\sqrt{2}$=.7071 inches.

$.7071\times 14=9.8994$ inches. $9.8994+2=11.8994$ inches.

$12-11.8994=.1006$ inch to spare.

Also solved by *CHARLES C. CROSS.*

121. Proposed by PAUL ROULET, A. M., Professor of Mathematics, Fairmount College, Wichita, Kas.

Three men, named Adams, Morris, and Stoughton, with their sons, Edward, Nathan, and Walter, have each a piece of land in the form of a square. Mr. Adams' piece is 23 rods longer on each side than Nathan's, and Mr. Stoughton's piece is 11 rods longer on each side than Edward's. Each father has 63 square rods of land more than his son. Which of these persons is father and son, respectively?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let the initial letter of person's name represent the length of the respective person's square piece of land.

Then $63+E^2=\square$, $63+N^2=\square$, and $63+W^2=\square$.

We are now to find which of these squares equals, respectively, A^2, M^2, and S^2.

The basis of work is the identity,

$$mn+\left(\frac{m-n}{2}\right)^2=\left(\frac{m+n}{2}\right)^2,$$

where $mn=63=9\times 7=21\times 3=63\times 1$.

From these sets of factors of 63, we obtain

$63+1^2=8^2$, $63+9^2=12^2$, and $63+31^2=32^2$.

Hence 1^2, 9^2, and 31^2 are the pieces of land belonging to the sons, and 8^2, 12^2, and 32^2 those belonging to their respective fathers.

But, from conditions of problem, we have $A-23=N$, and $S-11=E$.

Whence, it is evident that A must equal 32, and S must equal 12, from which we find $N=9$, and $E=1$.

Therefore, of the two remaining values, $M=3$, and $W=31$.

Now, since each father has 63 square rods of land more than his son, the man having 8^2 is the father of the son having 1^2; the man having 12^2 is the father of the son having 9^2; and so on.

∴ Mr. Morris is Edward's father; Mr. Stoughton is Nathan's father; and Mr. Adams is Walter's father.

A similar problem, said to be published in several newspapers of recent issue, is as follows: "Three Dutchmen, Hans, Klaus, and Hendricks, went to market to buy hogs, and took their wives with them. The names of the wives were Gertrude, Anna, and Katrine; but it was not known which was the wife of each man. They each, men and wives, bought as many hogs as each paid shillings, respectively, for each hog; and each man spent three guineas more than his wife. Hendricks bought 23 hogs more than Gertrude, and Klaus bought 11 more than Katrine. What was the name of each man's wife?"

A solution, similar to the above, gives Katrine as Hans's wife, Gertrude as Klaus's wife, and Anna as Hendricks's wife.

II. Solution by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

It is shown in geometry that the difference between two square areas is equal to a rectangle whose base is the sum of the sides of the squares and whose altitude is the difference between the sides.

63 square rods is the area of this rectangle. Then 63 is the product of two factors one of which is the sum and the other the difference of the sides of the pieces of father and son.

$63=63\times1$ or 21×3 or 9×7.

The side of the greater square, or the father's, is obtained by adding the sum and difference and dividing by two, and of the smaller, or son's, by subtracting the difference from the sum and dividing by two.

This process applied to each of the above pairs of factors gives for the sides of the pieces of father and son, respectively, 32 and 31, 12 and 9, 8 and 1. Since 31 and 9 differ by 23, 32 is the side of Adams's piece, and 9 is the side of Nathan's piece. Also 12 and 1 differ by 11. Therefore 12 is the side of Stoughton's piece, and 1 is the side of Edward's. This leaves 8 for the side of Morris's piece, and 31 for the side of Walter's.

Therefore the boys' names are Edward Morris, Nathan Stoughton, and Walter Adams.

III. Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

I. 1. Nathan is not the son of Mr. Adams, since the former has more than 63 square rods of land in excess of the latter.

Therefore either Edward or Walter is the son of Mr. Adams.

2. Edward is not the son of Mr. Stoughton, since the former has more than 63 square rods of land in excess of the latter.

Therefore Nathan or Walter is the son of Mr. Stoughton.

3. Walter is not the son of Mr. Morris, for then would Mr. Adams and Mr. Stoughton each have more land than the other, which is absurd. For then would Edward be the son of Mr. Adams, and Nathan of Mr. Stoughton; but Mr. Adams and Mr. Stoughton have more than 63 square rods, respectively, than Nathan and Edward, and, hence, more than each other.

Therefore either Nathan is the son of Mr. Morris, Walter of Mr. Stoughton, and Edward of Mr. Adams, or Edward is the son of Mr. Morris, Walter of Mr. Adams, and Nathan of Mr. Stoughton.

5. This indefinite solution suggests the idea that the proposer intended as a condition what is neither expressly stated nor necessarily implied in the problem, viz., that all the numbers are integral.

The second solution is on this assumption.

II. 1. Let x stand for the side, in rods, of any square, and $x+y$ for the side of a square whose length is y rods longer. Then $2xy+y^2=63$, whence $x=(63/2y)-\frac{1}{2}y$.

Now, the only positive integral values of y which will make x a positive integer are 1, 3, and 7, the values of x corresponding to which are, respectively, 31, 9, and 1. Hence, the pairs of squares differing by 63 square rods are as follows: (1) Sides, 32 rods and 31 rods; (2) 12 rods and 9 rods; (3) 8 rods and 1 rod.

2. Now, applying the first condition of the problem, it is evident that Mr. Adams owns the largest square, and that his son is Walter, that Nathan is the son of Mr. Stoughton, and Edward of Mr. Morris.

IV. Solution by SYLVESTER ROBINS, North Branch Depot, N. J.

$63=63\times1=21\times3=9\times7=32^2-31^2=12^2-9^2=8^2-1^2$.

$32-9=23$.

32^2 belongs to father Adams. 31^2 belongs to Mr. Adams's son.

9^2 belongs to Nathan. 12^2 to Nathan's father.

$12-1=11$.

The 12^2 belongs to father Stoughton. 1^2 belongs to Edward. 9^2 belongs to Nathan Stoughton.

The 8^2 belongs to father Morris. 1^2 to Edward Morris, and the 31^2 to Walter Adams.

Also solved by *COOPER D. SCHMITT, J. SCHEFFER*, and *G. B. M. ZERR*.

NOTE. Professor Scheffer sent in solutions of problems 118 and 119, and Professor Ellwood of problem 119. In the published solutions of No. 119, the following errors occur in the statement: For $1.\frac{9}{.100}$, read $1.\frac{9}{.001}$; line 8 from the bottom, for "$9001, the selling price," read "$9001, the cost price;" at top of page 270, for $1+\frac{9}{.009}$, read $1+\frac{9}{.001}$.

ALGEBRA.

97. Proposed by F. M. SHIELDS, Coopwood, Miss.

A farmer had 2080 pounds of grain at the depot, and gave a wagoner .75 cents per 100 pounds to haul it, paying him in the *same* grain at the following prices, viz.: 3-10 of the hauling bill was paid in corn at .58 cents per bushel of 56 pounds; 3-5 was paid in wheat at 1.55 cents per bushel of 60 pounds, and the balance of the bill was paid in oats at .36 cents per bushel of 32 pounds, the wagoner not charging for hauling his own grain. The load being delivered, how many bushels of each kind of grain did the wagoner get, and how many bushels of each kind did the farmer have left after paying the wagoner?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and C. B. GOULD, Colorado College, Colorado Springs, Col.

Let x=the hauling bill.

$x\div.75=\frac{4}{3}x$=number of hundredweight hauled.

$\therefore$ $400x/3$=number of pounds hauled.

$(\frac{3}{10}x\div.58)56=\frac{3}{10}x\times\frac{56}{29}\times\frac{56}{1}=840x/29$, pounds corn received by wagoner.

$(\frac{6}{10}x\div1.55)60=\frac{6}{10}x\times\frac{60}{31}\times\frac{60}{1}=720x/31$, pounds wheat received by wagoner.

$(\frac{1}{10}x\div.36)32=\frac{1}{10}x\times\frac{25}{9}\times\frac{32}{1}=80x/9$, pounds oats received by wagoner.

$\therefore$ $2080-400x/3=840x/29+720x/31+80x/9$.

$\therefore$ x=$10.698843, hauling bill.

$3x/5.80=5.5338843$ bushels of corn wagoner received.

$6x/15.50=4.1414877$ bushels of wheat wagoner received.

$x/3.60=2.9719008$ bushels of oats wagoner received.

$400x/3=1426.5124$ pounds hauled for farmer.

$\frac{3}{10}$ of $400x/3=40x$, $\frac{6}{10}$ of $400x/3=80x$, $\frac{1}{10}$ of $400x/3=40x/3$.

$40x\div56=7.6420307$ bushels of corn farmer had left.

$80x\div60=14.2651240$ bushels of wheat farmer had left.

$40x/3\div32=4.45785125$ bushels oats farmer had left.

II. Solution by J. D. CRAIG, Frankfort, N. J., and P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

Let $40x$=number of pounds of grain farmer retained.

Then $\frac{3}{4}\times40x=30x$=cents paid for carting.

$\frac{3}{10}$ of $30x=9x$=cents paid for carting corn.

$\frac{3}{5}$ of $30x=18x$=cents paid for carting wheat.

$\frac{1}{10}$ of $30x=3x$=cents paid for carting oats.

$9x\div\frac{3}{4}=12x$=pounds of corn farmer retained.

$18x\div\frac{3}{4}=24x$=pounds of wheat farmer retained.

$3x \div \frac{3}{4} = 4x$ = pounds of oats farmer retained.

$9x/58 \times 56 = \frac{252}{29}x$ = pounds of corn wagoner received.

$18x/155 \times 60 = \frac{216}{31}x$ = pounds of wheat wagoner received.

$3x/36 \times 32 = \frac{8}{3}x$ = pounds of oats wagoner received.

$40x + \frac{252}{29}x + \frac{216}{31}x + \frac{8}{3}x = 2080$ pounds of grain.....(1).

Solving, $x = 35\frac{401}{605}$.

Therefore, $12x = 427\frac{577}{605}$ pounds of corn retained by farmer.

$24x = 855\frac{549}{605}$ = pounds of wheat retained by farmer.

$4x = 142\frac{394}{605}$ = pounds of oats retained by farmer.

$\frac{252}{29}x = 309\frac{543}{605}$ = pounds of corn paid wagoner.

$\frac{216}{31}x = 248\frac{296}{605}$ = pounds of wheat paid wagoner.

$\frac{8}{3}x = 95\frac{61}{605}$ = pounds of oats paid wagoner.

Reducing to bushels we find that the farmer retains $7\frac{1469}{4235}$ bushels corn, $14\frac{8025}{30225}$ bushels of wheat, and $4\frac{2770}{6049}$ bushels of oats.

The wagoner gets $5\frac{60190}{169400}$ bushels corn, $4\frac{4285}{30225}$ bushels wheat, and $2\frac{588}{6049}$ bushels oats.

III. Solution by B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.; ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.; and O. S. WESTCOTT, Principal North Division High School, Chicago, Ill.

Let x, y, and z equal, respectively, the number of pounds of corn, wheat, and oats that the farmer had left. Then from the condition of problem we have

$$\frac{3}{10}\left[\frac{.75}{.58}\,\frac{(x+y+z)}{100}\right]56 = \frac{x}{100} \times \frac{.75}{.58} \times 56,$$

or, by reducing, we have $7x - 3y + 3z = 0$.....(1), and in like manner we get $3x - 2y + 3z = 0$.....(2), and $x + y - 9z = 0$ .....(3), from which we get $y = 2x = 6z$.

$\therefore\ x + y + z = 10z$.

$.75 . \frac{z}{100} . \frac{32}{.36} = \frac{8}{3}z$ = pounds of oats wagoner received, and in like manner $\frac{54}{31}z$ and $\frac{63}{29}z$ equal pounds of wheat and corn that the wagoner received for hauling.

$\therefore$ we have $2080 - 10z = \frac{63}{29}z + \frac{54}{31}z + \frac{8}{3}z$.

$\therefore\ z = 142\frac{5122}{78865}$ pounds or $4\frac{3601}{18865}$ bushels of oats farmer received.

$y = 6z = 855\frac{7137}{78865}$ pounds or $14\frac{6255610}{3595950}$ bushels of wheat farmer received.

$x = 3z = 427\frac{7561}{78865}$ pounds or $7\frac{3533477}{3595955}$ bushels of corn farmer received.

$\therefore$ If the wagoner received 75 cents per 100 pounds for hauling, he received—

$\frac{74796\,8}{78865}$ pounds or $2\frac{76445}{79655}$ bushels of oats,

$\frac{24373\,44}{78865}$ pounds or $5\frac{41999}{78865}$ bushels of corn,

$\frac{19543\,68}{78865}$ pounds or $4\frac{11128}{786550}$ bushels of wheat.

We understand the problem to mean that the wagoner received corn for hauling corn, wheat for wheat, etc.

CALCULUS.

NOTE ON PROBLEM 84, BY DR. E. WOELFFING, STUTTGART, GERMANY.

The solution of question 84 can be found in some theorems proved by W. Merkelbach (*Ueber Rollkurven welche von einer Graden eingehüllt werden*, Diss. Marburg, 1881). The first of them is the following:

If a curve, C, rolls upon another curve, C', and a point, P, in the plane of C describes a straight line L, and we make afterwards the curve C' roll upon the curve C, then a straight line, L, in the plane of C' will always pass through P (page 18 of the paper quoted).

Now (and this is the second theorem of Merkelbach) if a sinusoid rolls upon an ellipse, a straight line in the plane of the former passes through a focus of the latter (page 24); therefore, if the ellipse rolls upon the sinusoid, any one of the foci of the former will describe a straight line.

Stuttgart, Germany, July 19, 1899.

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91. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Find the area of a loop of the curve $r^2\cos\theta=a^2\sin3\theta$. [From Hall's *Differential and Integral Calculus*].

I. Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Ind.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; ELMER SCHUYLER, Reading, Penna.; and J. SCHEFFER, A. M., Hagerstown, Md.

The curve has two equal loops, one in the first and the other in the third quadrant.

The limits of θ are 0 and $\frac{1}{3}\pi$.

$$\therefore A=\tfrac{1}{2}\int_0^{\frac{1}{3}\pi} r^2d\theta=\tfrac{1}{2}a^2\int_0^{\frac{1}{3}\pi}\frac{\sin3\theta d\theta}{\cos\theta}$$

$$=\tfrac{1}{2}a^2\int_0^{\frac{1}{3}\pi}(4\sin\theta\cos\theta-\tan\theta)d\theta=\tfrac{1}{2}a^2(3-2\log2).$$

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.

The shape of the curve is seen in the diagram.

The limits are evidently from 0° to 30° for a loop.

Then $A=\frac{1}{2}\int r^2d\theta=\frac{1}{2}\int_0^{\frac{1}{6}\pi}\frac{a^2\sin3\theta}{\cos\theta}d\theta$

$$=\frac{a^2}{2}\int_0^{\frac{1}{6}\pi}\frac{3\sin\theta-4\sin^3\theta}{\cos\theta}d\theta=\frac{a^2}{2}\int_0^{\frac{1}{6}\pi}3\tan\theta d\theta$$

$$-2a^2\int_0^{\frac{1}{6}\pi}(\tan\theta-\sin\theta\cos\theta)d\theta$$

$$= \frac{3a^2}{2}\log\sec\theta - 2a^2\log\sec\theta + a^2\sin^2\theta\Big]_0^{\frac{1}{3}\pi} = a^2\sin^2\theta - \frac{a^2}{2}\log\sec\theta\Big]_0^{\frac{1}{3}\pi}$$

$$= \frac{3a^2}{4} - \frac{a^2}{2}\log 2.$$

[NOTE. In the figure, the loop in the fourth quadrant should be in the first, and the one in the second in the third.

92. Proposed by B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.

How much wood is taken from a log 12 inches in diameter, by boring a two-inch hole through the center, the axis of the hole being perpendicular to axis of log ?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J SCHEFFER, A. M., Hagerstown, Md., and WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

$$V = 8\int\int\int dxdydz = \text{volume}.$$

The equation of the surface of the cylinder corresponding to the log is

$$x^2 + y^2 = 36 = R^2,$$

and the equation of the surface of the cylinder corresponding to the auger-hole is

$$x^2 + z^2 = 1 = r^2.$$

$$V = 8\int\int\int ydxdz = 8\int_o^r\int_0^{\sqrt{(r^2-x^2)}} \sqrt{(R^2-x^2)}\,dxdy.$$

$$\therefore\ V = 8\int_o^r \sqrt{[(r^2-x^2)(R^2-x^2)]}\,dx$$

$$= \tfrac{8}{3}R^3\{[1+(r/R)^2]E(r/R) - [1-(r/R)^2]F(r/R)\}$$

$$= 576\{(1+\tfrac{1}{3\,6})E(\tfrac{1}{6}) - (1-\tfrac{1}{3\,6})F(\tfrac{1}{6})\}$$

$$= 576\{\tfrac{3\,7}{3\,6}E(\tfrac{1}{6}) - \tfrac{3\,5}{3\,6}F(\tfrac{1}{6})\} \text{ cubic inches.}$$

Also $V = 8\int_o^r \sqrt{(r^2-x^2)}\left[R - \frac{x^2}{2R} - \frac{x^4}{8R^3} - \ldots - \frac{1.3.5.7\ldots.(2n-1)x^{2n}}{2.4.6\ldots.2nR^{2n-1}}\right]dx.$

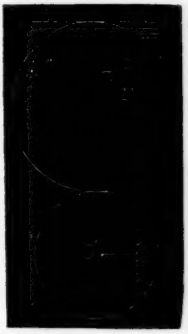
$$\text{Now } \int_o^r \sqrt{(r^2-x^2)}\,x^{2n}dx = \frac{1.3.5.7\ldots.(2n-1)}{2.4.6.8\ldots.(2n+2)}\pi r^{n+2}.$$

$$\therefore\ V = 4r^2R\pi\left[\tfrac{1}{2} - \frac{r^2}{16R^2} - \frac{r^4}{128R^4} - \ldots - \frac{(1.3.5.7\ldots.\,2n-3)^2(2n-1)r^{2n}}{(2.4.6\ldots.\ldots.\,2n)^2(2n+2)R^{2n}} - \ldots\right]$$

$$\therefore\ V = 24\pi(\tfrac{1}{2} - \tfrac{1}{5\,7\,6} - \tfrac{1}{1\,6\,5\,8\,8\,8} - \tfrac{5}{9\,5\,5\,5\,1\,4\,8\,8} - \ldots -) = 37.56784 \text{ cubic inches.}$$

II. Solution by GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

This is easily seen to depend on the finding of the volume enclosed between the larger cylinder and a tangent plane, and the smaller cylinder.



Let r=radius of large cylinder.

a=radius of small cylinder.

z=distance of cutting plane from axis of the large cylinder.

We have then to evaluate the integrals

$$a^2\int_{\sqrt{(r^2-a^2)}}^{r}\cos^{-1}\frac{\sqrt{(r^2-z^2)}}{a}dz-\int_{\sqrt{(r^2-a^2)}}^{r}(r^2-z^2)\sqrt{(a^2-r^2+z^2)}dz.$$

Putting $\frac{\sqrt{(r^2-z^2)}}{a}=\sin\theta$, the first integral becomes

$$-\frac{\pi a^4}{2}\int\frac{\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}+a^4\int\frac{\theta\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}.$$

The second becomes $-a^4\int\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}$.

$$\int\frac{\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}=-\frac{\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}.$$

Combining the second integral with the latter part of the first, we have

$$\int(\theta+\sin\theta\cos\theta)\frac{\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}$$

which integrated by parts gives

$$\frac{-(\theta+\sin\theta\cos\theta)\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}+\int\frac{2\cos^2\theta\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}d\theta.$$

The last integral

$$=2\int\frac{\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}d\theta-2\int\sin\theta\frac{\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}d\theta.$$

We have then

$$\tfrac{1}{2}\text{ Volume}=\left[\frac{\pi a^2}{2}\sqrt{(r^2-a^2\sin^2\theta)}-a^2(\theta+\sin\theta\cos\theta)\sqrt{(r^2-a^2\sin^2\theta)}\right]_0^{\frac{1}{2}\pi}$$

$$+2a^2\int \sqrt{(r^2-a^2\sin^2\theta)}d\theta - 2a^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\sqrt{(r^2-a^2\sin^2\theta)}d\theta.$$

When $\theta=0$, the first line is $\frac{1}{2}(\pi a^2 r)$.

When $\theta=\frac{1}{2}\pi$, its value is 0.

To evaluate $\int \sqrt{(r^2-a^2\sin^2\theta)}d\theta$, put $a=re$, expand by binomial formula, and integrate the terms.

$$(1-e^2\sin^2\theta)^{\frac{1}{2}} = 1-\tfrac{1}{2}e^2\sin^2\theta-\tfrac{1}{8}e^4\sin^4\theta-\tfrac{1}{16}e^6\sin^6\theta-\ldots\ldots$$

Then $\int_0^{\frac{1}{2}\pi}\sqrt{(r^2-a^2\sin^2\theta)}d\theta=\frac{1}{2}(\pi r)(1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6\ldots\ldots)$

The remaining integral treated in the same way gives

$$\tfrac{1}{2}\pi r(\tfrac{1}{2}-\tfrac{3}{16}e^2-\tfrac{15}{128}e^4-\tfrac{35}{256}\tfrac{5}{8}e^6-\ldots\ldots$$

$$\text{Finally, Volume}=\frac{\pi a^2 r}{8}\left(e^2+\tfrac{1}{8}e^4+\frac{5e^6}{128}+\ \ldots\ \right)$$

In our example, $e=\frac{1}{6}$, $a=1$, $r=6$.

$$\text{Hence, } V=\frac{9\pi}{2}\left(\frac{1}{36}+\frac{1}{8\times 1296}+\frac{5}{128\times 6^6}+\ldots\ldots\right)=\tfrac{1}{8}\pi(1+\tfrac{1}{288}) \text{ nearly.}$$

This subtracted from $\pi a^2 r$ and the result doubled gives the volume common to the two cylinders.

MECHANICS.

90. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

Adopting the hypothesis that the planets were originally all one mass revolving about a fixed center and were formed by an explosion of this mass at some point in its path; prove that, if the law of nature were that force varies directly as the distance, the planets would all have collided again simultaneously, and find an expression for the time between the explosion and collision.

Solution by the PROPOSER.

Regarding the original mass as a particle, the pieces after explosion, no matter what their initial velocities or directions, would all move in concentric ellipses; and as their paths intersect in one point, viz., the position of the original mass at the moment of explosion, they must all have another point in common at the extremity of the common diameter through the first common point.

We have for the equations of motion for any piece

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0 \quad \ldots\ldots(1). \qquad m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]=-r \ldots\ldots(2).$$

From (1) by integration,

$$r^2\frac{d\theta}{dt}=h \ldots\ldots(3),$$

a constant depending on the initial velocity and angle of projection.

To integrate (2) let $u=1/r$. Then

$$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}=-h\frac{du}{dt} \text{ from (3).}$$

$$\therefore \frac{d^2r}{dt^2}=-h\frac{d}{dt}\left(\frac{du}{d\theta}\right)=-h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting these results in (2) and using (3) we have

$$\frac{d^2u}{d\theta^2}=-u+\frac{k}{h^2u^3} \ldots\ldots(4),$$

where k is a constant depending on the force of attraction.

Multiply by $2\frac{du}{d\theta}$ and integrate and we have

$$\left(\frac{du}{d\theta}\right)^2=-u^2-\frac{k}{h^2u^2}+c_1 \ldots\ldots(5),$$

$$\theta=\int\frac{hudu}{\sqrt{(c_1h^2u^2-h^2u^4-k)}}=\tfrac{1}{2}\cos^{-1}\left(\frac{2hu^2-c_1h}{\sqrt{(c_1^2h_2-4k)}}\right)+c_2.$$

Hence simplifying and restoring value of u we get for the equation of the path

$$\frac{1}{r^2}=\frac{c_1}{2}-\frac{\sqrt{(c_1^2h^2-4k)}}{2h}+\frac{\sqrt{(c_1^2h^2-4k)}}{h}\cos^2(\theta-c_2).$$

Transforming to rectangular coödinates this becomes

$$\frac{x^2}{\frac{2h}{c_1h+\sqrt{(c_1^2h^2-4k)}}}+\frac{y^2}{\frac{2h}{c_1h-\sqrt{(c_1^2h^2-4k)}}}=1.$$

Let A be area of the ellipse, and a, b, its semi-axes. Then

$$A=\pi ab=\pi\times\frac{2h}{c_1h+\sqrt{(c_1^2h_2-4k)}}\times\frac{2h}{c_1h-\sqrt{(c_1^2h_2-4k)}}=\frac{\pi h}{\sqrt{k}}.$$

For the area of any curve, we have from the calculus

$$A=\tfrac{1}{2}\int_{\theta_0}^{\theta_1}r^2d\theta.$$

If we regard the angle θ_0 as the initial angle it is 0, and as θ is a function of the time, this integral may be written

$$A=\tfrac{1}{2}\int_0^{t_1}r^2d\theta=\tfrac{1}{2}\int_0^t r^2\frac{d\theta}{dt}dt=\tfrac{1}{2}h\int_0^{t_1}dt=\tfrac{1}{2}ht \text{ by (3)}.$$

$\therefore t=2A/h$. That is, the periodic time always equals twice the area swept over by the radius vector divided by the constant h.

Hence for the given ellipse we have

$$t=\frac{2\pi h/\sqrt{k}}{h}=\frac{2\pi}{\sqrt{k}}.$$

Therefore the time it would require for a piece to travel from the point of explosion to the next point of intersection is $\frac{1}{2}t-\pi/\sqrt{k}$, and as k is a constant the same for all pieces, we see this time is the same for all, and hence they must collide simultaneously.

91. Proposed by CHARLES C. CROSS, Whaleyville, Va.

The bow of a boat which is a inches wide is inclined at an angle α. When in motion in perfectly calm water the water was found to rise b inches on the bow. Required the velocity of the boat.

No solution of this problem has been received.

92. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A particle, starting at the vertex, slides down a smooth parabolic curve. Find the initial velocity of the particle so that it may leave the curve at the extremity of semi-latus rectum.

Solution by GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Take the vertex as origin and positive x downwards.

The equation of the curve is $y^2=2px$.

Let N be the normal reaction, v the velocity, θ the angle between normal and x-axis, m the mass of particle, ρ the radius of curvature.

Then $N=mv^2/\rho+mg\cos\theta$.

$$\rho=\frac{(y^2+p^2)^{\frac{3}{2}}}{p^2},\quad v^2=v_0^2+2gx,\quad \cos\theta=\frac{p}{\sqrt{(y^2+p^2)}}.$$

Accordingly, $N=mg(p-\frac{v_0^2}{g})=\frac{m}{p}(pg-v^2)$.

If $v_0^2=pg$ the normal reaction is always zero.
If $v_0^2>pg$ the particle leaves the curve at once.
If $v_0^2<pg$ the particle is constrained to move on the curve.
Then the particle either—
1°. Describes the curve freely without leaving it; or,
2°. Leaves the curve at the beginning of the motion; or,
3°. Describes the curve under constraint without leaving it.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

DIOPHANTINE ANALYSIS.

75. Proposed by CHARLES CARROLL CROSS, Whaleyville, Va.

Arrange the consecutive integers 1 to n^2 as a magic square, where n is odd. Apply when $n=9$.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

This solution for odd magic squares is adapted from a similar solution found in an article on "Evening Entertainments."

We divide a large square into as many little spaces as there are numbers in the magic square.

We conceive a similar large square, similarly divided, bordering on the right, another on the bottom, another on the left, and a fourth meeting the right bottom corner of the given square.

We fill in the numbers consecutively, commencing with unity, *from left diagonally down to right, beginning with the space immediately below the center space.*

When reaching the limit of the given square, we continue the next number into the "bordering square." This number will then be placed in the *corresponding space of the given square.*

When, in filling in the numbers, we meet a space that is already occupied, we take the space *next diagonally down left from this occupied space*, and continue as before.

Take a magic square having 9 numbers on a side. There will then be 9^2, or 81 spaces in the square, to be filled with the numbers 1 to 81 inclusive.

Beginning with the space next below the center space, and filling in diagonally down to right, we find 4 reaches the limit of the square. Whence 5 occupies, in the "bordering square," the right upper corner. We now put 5 in the right upper corner of the given square, and find it at the limit of the square. Whence 6 occupies, in the "bordering square," the space next below the left upper corner. We now place 6 in the corresponding space of the given square, and proceed 6, 7, 8, 9, when we meet the space occupied by 1. We then put 10 in the space next diagonally down left from 1, and proceed 10, 11, 12, etc., until all the spaces are filled.

		37	78	29	70	21	62	13	54	5		
	46	6	38	79	30	71	22	63	14	46	6	
		47	7	39	80	31	72	23	55	15	47	
		16	48	8	40	81	32	64	24	56	16	
		57	17	49	9	41	73	33	65	25	57	
		26	58	18	50	1	42	74	34	66	26	
		67	27	59	10	51	2	43	75	35	67	
		36	68	19	60	11	52	3	44	76	36	
		77	28	69	20	61	12	53	4	45	77	
		37	78	29	70	21	62	13	54	5	46	

AVERAGE AND PROBABILITY.

79. Proposed by the late ENOCH BEERY SEITZ.

Two equal spheres touch each other externally. If a point be taken at random within each sphere, show that (1) the chance that the distance between the points is less than the diameter of either sphere is 13/35, and (2) the average distance between them is $11/5r$. [This is problem 5835, *Educational Times*, of London.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(1) Let A, B be the centers and C the point of contact of the two spheres, each radius r.

From any point P in DC with a radius$=2r$ describe a sphere cutting B in Q, R. From B as a center with a radius BP describe a sphere cutting A in K, M. If P is the first point, the second point must fall within the double-convex lens $CQRC$. P may fall anywhere on the zone KPM and the second point must fall in a section of B equal to the double-convex lens $CQRC$.

From P as center with a radius $PS<2r$ but$>PC$, draw the zone SLT. Let $DP=x$, $PS=y$, area of zone $KPM=2\pi.BP.PG$, area of zone $SLT=2\pi.PS.HL$.

$BP=3r-x$, $AG=r-x-PG$, $BG=3r-x-PG$, $PS=y$, $BH=3r-x-y+HL$, $PH=y-HL$.

$KG^2=r^2-(r-x-PG)^2=(3r-x)^2-(3r-x-PG)^2$.

$\therefore$ $PG=x(2r-x)/4r$.

$SH^2=r^2-(3r-x-y+HL)^2=y^2-(y-HL)^2$.

$\therefore$ $HL=[r^2-(3r-x-y)^2]/2(3r-x)$.

$\therefore$ Area of zone $KPM=(\pi x/2r)(3r-x)(2r-x)$.

. Area of zone $SLT=[\pi y/(3r-x)][r^2-(3r-x-y)^2]$.

Let p=chance, $\triangle$=average distance.

$$\therefore\ p=\{\pi^2/[2r(\tfrac{4}{3}\pi r^3)^2]\}\int_0^{2r}x(3r-x)(2r-x)dx\int_{2r-x}^{2r}[y/(3r-x)][r^2-(3r-x-y)^2]dy$$

$$=(3/128r^7)\int_0^{2r}(14rx^5-x^6-48r^2x^4+48r^3x^3)dx=(3/128r^7)(1664r^7/105)=\tfrac{13}{35}.$$

$$2.\ \triangle=\{\pi^2/[2r(\tfrac{4}{3}\pi r^3)^2]\}\int_0^{2r}x(3r-x)(2r-x)dx\int_{2r-x}^{4r-x}[y^2/(3r-x)][r^2-(3r-x-y)^2]dy$$

$$=(3/40r^4)\int_0^{2r}(92r^3x-106r^2x^2+40rx^3-5x^4)dx=(3/40r^4)(88r^5/3)=11r/5.$$

80. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A box contains 100 balls marked from 1 to 100; 13 balls are drawn at random. What is the chance that the balls marked from 1 to 10 are included in the 13 drawn?

Solution by J. W. YOUNG, Columbus, Ohio.

Since in all the favorable chances only three balls may vary, the total number of favorable chances is ${}^{90}C_3$, *i. e.*, the number of combinations of 90 things taken 3 at a time.

The total number of ways in which the balls may be drawn is, of course, ${}^{100}C_{13}$.

Hence the desired probability is equal to

$$\frac{{}^{90}C_3}{{}^{100}C_{13}}=\frac{\dfrac{90.89.88}{1.2.3}}{\dfrac{100.99.98.97.96.....89.88}{1.2.3.5.....13}}=\frac{1}{67515927540}.$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

124. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

At what time between 5 and 6 o'clock is the minute hand midway between 12 and the hour hand? When is the hour hand midway between 4 and the minute hand?

125. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

A Quaker once, we understand
For his three sons laid off his land,
And made three equal circles meet
So as to bound an acre neat.
Now in the center of the acre,
Was found the dwelling of the Quaker;
In centers of the circles round,
A dwelling for each son was found.
Now can you tell by skill or art
How many rods they live apart?

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

ALGEBRA.

115. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

Find the conditions of the coefficients of a general biquadratic equation so that it may be solved by quadratics.

116. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equations:

$$w(xy+xz+yz)=a;\ x(wy+wz+yz)=b;$$
$$y(wx+wz+xz)=c;\ z(wx+wy+xy)=d.$$

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

GEOMETRY.

135. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If a hyperbola be described touching the four sides of a quadrilateral which is inscribed in a circle, and one focus lie on the circle, the other focus will also lie on the circle.

136. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.

Construct a triangle having given the base, the median line to the base, and the difference of the base angles.

137. Proposed by J. W. YOUNG, Fellow and Assistant, Ohio State University, Columbus, O.

A right cone has its vertex in a horizontal plane, its axis being perpendicular to the plane. A string has one extremity attached to a point on the cone. The other extremity, P, of the string is kept in the plane, and the string is then wound around the cone, without being allowed to slip. Show that the spiral generated by P cuts all straight lines through the vertex at the same angle.

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

CALCULUS.

106. Proposed by M. C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Ind.

$$\int_0^\pi \frac{\cos rx\,dx}{1-2a\cos x+a^2}=\frac{\pi r^2}{1-a^2}.$$

[*Williamson's Integral Calculus*, 6th Edition, page 174.]

107. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

The speed of signaling in submarine telegraph-cable varies as $x^2\log(1/x)$, in which x is the ratio of the radius of the core to that of the covering. Prove that the *maximum speed* is attained when this ratio is $1:\sqrt{e}$.

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

MECHANICS.

101. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

Find the center of gravity of a cone that has a specific gravity of 1 (one) at the top and 2 (two) at the base.

102. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A heavy particle with a light string attached is placed on the edge of a smooth table. A boy, holding the string horizontally, runs at right angles to the string. Determine the motion of the particle (1) when the boy runs with a uniform velocity; (2) when he runs with a uniform acceleration.

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

DIOPHANTINE ANALYSIS.

83. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find three numbers in arithemetical progression whose sum is a square and cube.

84. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

The nth term of an infinite series of "nests" contains all the prime, integral, rational parallelopipeds that have 3^n for their solid diagonals. It is required to determine the general expression for N=the number of such solids in nth term, and to exhibit the dimensions of all the "eggs" in the first six nests.

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

AVERAGE AND PROBABILITY.

88. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the average volume of the tetrahedron formed by joining four random points in a sphere.

89. Proposed by B. F. FINKEL, A.M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

An inch auger-hole is bored at random through a six-inch sphere. Find the average volume of the auger-hole.

90. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

During a heavy rain storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond? [From *Byerly's Integral Calculus.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

MISCELLANEOUS.

85. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Prove that at least one of the three sides of a rational right triangle must be divisible by 5.

86. Proposed by GEORGE LILLEY, Ph.D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent., annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the payment on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$?

87. Proposed by A. H. HOLMES, Brunswick, Me.

Find $f(x)$ from $f(x+1)-f^2(x)=x$.

NOTES.

Dr. E. M. Blake, formerly instructor in mathematics in Purdue University, has been appointed as Honorary Fellow in Mathematics in Cornell University.

The two books, Whitehead's Universal Algebra and Killing's Einführung in die Grundlagen der Geometrie, which were particularly signalized in Halsted's Report on Progress in Non-Euclidean Geometry, have been entered in competition for the Lobachevski Prize of 1900.

On the twenty-fourth of December, 1899, the Physico-Mathematic Society of Kazan (Russia) celebrated a Jubilee in honor of the twenty-fifth year of professional and scientific service of its President, Professor A. Vasiliev. It is also the fifteenth year of his presidency. Professor Vasiliev has been an extraordinarily important figure in Russian science. Outside of Russia he has chiefly been known for his remarkable discourse on Lobachevski, Englished by Halsted. A German translation of his book on Tchebychev is to be published this month by Teubner at Leipzig. The first volume of an edition of Tchebychev's Collected Works, in French, has just appeared, edited by the academicians Markof and Sonine. It contains a very fine portrait of the great mathematician and the first thirty-four Memoirs of Vasiliev's list.

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from December Number.]

CHAPTER V.

RELATIVE MULTIPLICATION.

56. DEFINITION.—In the preceding chapter no reference was made to the space in which the factors multiplied were contained. Now, in ordinary multiplication of geometrical magnitudes, there is a limit beyond which one can not go. For instance, when one has multiplied the length, breadth, and thickness together, he can add no other dimension. This suggests the idea of taking any arbitrary number as n for the number of dimensions of the space considered. In any investigation, then, what we will call "the space considered" is that space of the original units which contains all the quantities involved. Multiplications made with reference to the space considered are called *Relative Multiplications.*

57. DEFINITION.—If in a space of the nth order, the combinatory product of the original units $e_1 e_2 \ldots\ldots e_n$, is set equal to the scalar unity, and E is a unit of any order, (*i. e.* either one of the original units or a combinatory product of two or more of them), then the *complement of* E is $+E'$ or $-E'$, where E' is the combinatory product of all the units which do not appear in E. The complement of E is $+E'$ when $[EE']=+1$; and $-E'$ when $[EE']=-1$. Let the com-

plement of E be denoted by $|E$. This mark, the sign of the complement, in Grassmann is a vertical line somewhat longer that the caps and about as heavy as the vertical stroke in cap N. Then

$$|E=[EE']E'.$$

The end sought is to get $[E|E]=+1$. To show that this is attained, multiply the equation above through by E. Then

$$[E|E]=[EE'][EE']=+1,$$

since whether $[EE']=+1$ or -1, $[EE'][EE']=+1$.

The reason why we have this ambiguity of sign in the product $[EE']$ is because each original unit is allowed to have either the plus or minus sign.

In particular we have

$$|1=1, \text{ or } |\alpha=\alpha,$$

by multiplying the first equation through by α.

58. DEFINITION.—By the *Complement* of any quantity A will be understood that quantity $|A$ which is obtained by replacing each product of units in the derived expression for A by its complement. Expressing the same in a formula we have

$$|A\equiv|(\alpha E_1+\alpha_2 E_2+\ldots\ldots)=\alpha_1|E_1+\alpha_2|E_2+\ldots\ldots$$

where E_1, E_2, ... are products of units of any order.

59. *The complement of the complement of any quantity A is equal either to A, or to $-A$, according as $(-1)^{qr}$ is $+$ or $-$*, where q and r are the orders of the quantity and its complement.

The Proof depends on 41.

60. *If the order of a space n is odd.* $\| A=A$ *: if n is even.* $\| A=(-1)^q A$, *where q is the order of A.*

PROOF.—By 59, $\| A=(-1)^{q(n-q)}A$. Then if n is odd, $q(n-q)$ is even, whether q be even or odd : but if n is even, then if q is even $q(n-q)$ is even, and if q is odd $q(n-q)$ is odd. The theorem as stated readily follows.

61. DEFINITION.—If the sum of the orders of two units is less than or equal to the order n of the space considered, then by their *progressive product* is understood their outer product (52), with the provision, however, that the product of the n original units is unity. On the other hand if the sum of the orders of two units is greater than the order n of the space, then by their *regressive product* is understood that quantity whose complement is the progressive product of the complements of these units.

Thus, if the sum of the orders of E and $F>n$, we have that